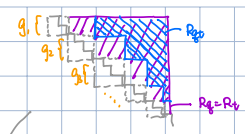
Theorem 3.5.1: Given a den $(h, p, \vec{d}, \vec{e})$ and $\vec{g} = (g_1, \dots, g_n)$ with $g_k = \sum_{j=0}^{k-1} (d_j - e_j)$, define a Catalanimal



$$H := H(R_{\vec{g}}, R_t, R_{\vec{g}t}, (\underbrace{d_0 - e_1, \dots, d_0 - e_1}_{g_1}, \underbrace{d_1 - e_2, \dots, d_1 - e_2}_{g_2}, \dots, \underbrace{d_{n-1} - e_n, \dots, d_{n-1} - e_n}_{g_n}))$$

of length $\ell = g_1 + \dots + g_n$, where
- $R_{\vec{g}} = R_t = \{\varepsilon_i - \varepsilon_j : i < j,\ i, j$ are in distinct blocks of the partition of $[\ell]$ into intervals of lengths $g_k\}$;
- $R_{\vec{g}t} = \{\varepsilon_i - \varepsilon_j \in R = R_t : i, j$ are in NON-adjacent blocks$\}$.

$$\underbrace{[1, g_1], [g_1+1, g_1+g_2], \dots, [g_1 + \dots + g_{n-1} + 1,\ \overset{\ell}{\overset{\shortparallel}{g_1 + \dots + g_n}}]}_{\ell}$$

Then the polynomial part of H is given by

$$H_{pol}(z) = \sum_{\pi : \text{nest}} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \mathcal{G}_{\nu(\pi)}(z_1, \dots, z_\ell; q^{-1}) \ . \quad \left(= \sum_{\pi : \text{nest}} \sum_{\substack{P : \text{positive} \\ \text{labelling of } \pi}} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi) - \mathrm{Inv}(P)} z^P \right)$$

ℓ variables

Corollary 3.5.2: Given a den $(h, p, \vec{d}, \vec{e})$, let $\mathcal{Y} = \Psi_q(H) \in \mathcal{E}^+$ be the Schiffmann algebra element represented by the tame Catalanimal H in Theorem 3.5.1. Then

$$\mathcal{Y} \cdot 1 = \sum_{\pi : \text{nest}} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1}) \quad \left(= \sum_{\pi : \text{nest}} \sum_{\substack{N : \text{negative} \\ \text{labelling of } \pi}} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi) - \mathrm{Inv}(N)} x^N \right)$$

Proof: By Prop. 2.3.2, $\omega(\mathcal{Y} \cdot 1) = H_{pol}$.
Hence by Theorem 3.5.1, $\mathcal{Y} \cdot 1 = \omega H_{pol} = \sum_{\pi : \text{nest}} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$. $\square$

Corollary 3.5.3: (i) Both $\sum_{\pi : \text{nest}} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \mathcal{G}_{\nu(\pi)}(z_1, \dots, z_\ell; q^{-1})$ and $\sum_{\pi : \text{nest}} t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$ are q, t-symmetric.

(ii) Both $H_{pol}(z)$ and $\mathcal{Y} \cdot 1$ are q, t-Schur positive over $\mathbb{N}[q, t]$.

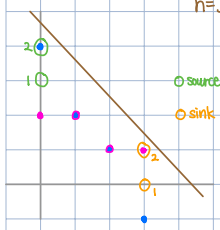
Proof: (i) The Catalanimal H constructed in Theorem 3.5.1 is q, t symmetric (and hence H_{pol})

(ii) The coeff. are in $\mathbb{Z}[q^{-1}, t]$ a priori.
By (i), the coeff. are in $\mathbb{Z}[q, t]$.
Since $\mathcal{G}_\nu(X; q)$ are q-Schur positive, the coeff are in $\mathbb{N}[q, t]$. $\square$

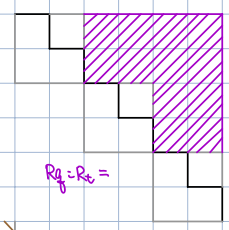
Remark: If the den has no nest, then Theorem 3.5.1 implies $H_{pol} = \mathcal{Y} \cdot 1 = 0$.

Examples on the next 2 pages.

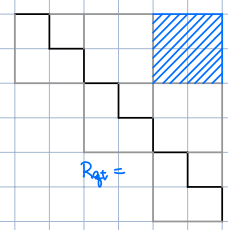
$n=3$, $p=1+\varepsilon$, $\vec{a}=(4,2,1,-1)$, $\vec{e}=(2,2,1,1)$, $\vec{g}=(2,2,2)$ #source = #sink = 2, $(d_0-e_1, d_1-e_2, d_2-e_3)=(2,1,0)$, $\ell=|\vec{g}|=6$



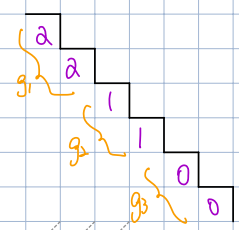
$\sigma=\omega_0$

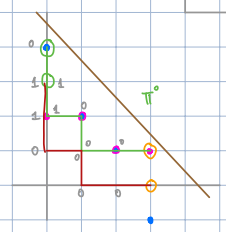


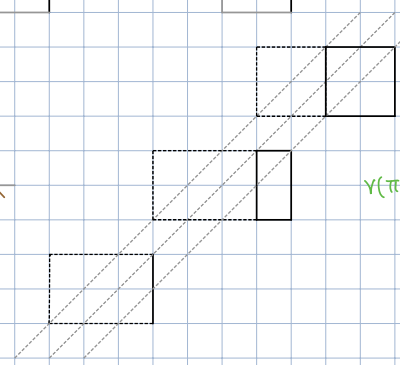
$R_{\vec{g}}=R_t=$



$R_{\vec{g}t}=$







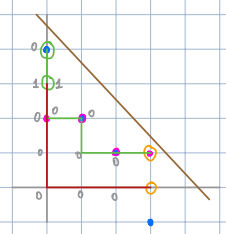
$\nu(\pi^0)=\left(\frac{(3,3)}{(3,3)}, \frac{(4,4)}{(3,3)}, \frac{(4,4)}{(2,2)}\right)$

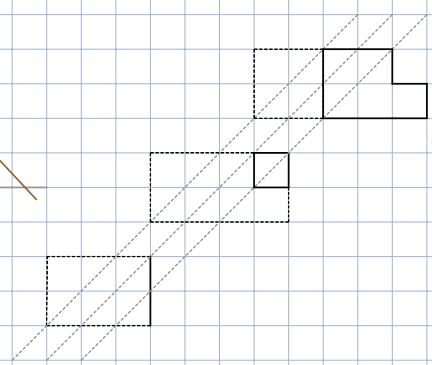
$a(\pi^0)=0$

$\mathrm{dinv}_p(\pi)=1+1+1+1=4$

$\lambda_{(1)}=\phi$

$\lambda_{(2)}=\phi$





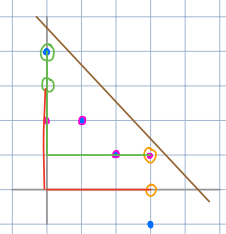
$\nu(\pi)=\left(\frac{(3,3)}{(3,3)}, \frac{(4,4)}{(4,3)}, \frac{(5,4)}{(2,2)}\right)$

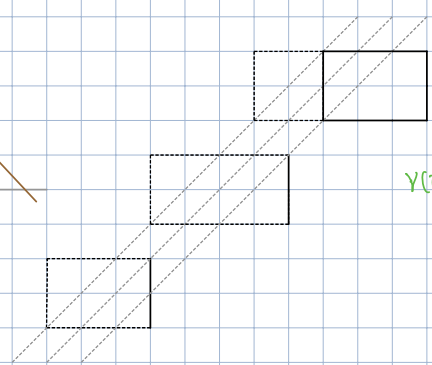
$a(\pi)=1$

$\mathrm{dinv}_p(\pi)=2$

$\lambda_{(1)}=\phi$

$\lambda_{(2)}=(1)$





$\nu(\pi)=\left(\frac{(3,3)}{(3,3)}, \frac{(4,4)}{(4,4)}, \frac{(5,5)}{(2,2)}\right)$

$a(\pi)=2$

$\mathrm{dinv}_p(\pi)=0$

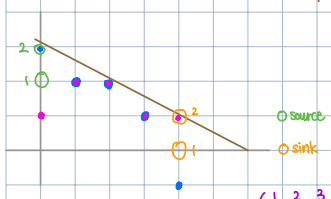
$\lambda_{(1)}=(1)$

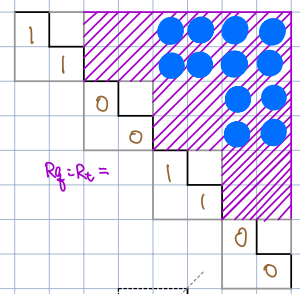
$\lambda_{(2)}=(1)$

$$H_{pol}(z_1,\dots,z_8)=q^4\,\mathcal{G}_{\left(\frac{(3,3)}{(3,3)},\frac{(4,4)}{(3,3)},\frac{(4,4)}{(2,2)}\right)}(x_1,\dots,x_8;q^{-1})+tq^2\,\mathcal{G}_{\left(\frac{(3,3)}{(3,3)},\frac{(4,4)}{(4,3)},\frac{(5,4)}{(2,2)}\right)}(x_1,\dots,x_8;q^{-1})+t^2\,\mathcal{G}_{\left(\frac{(3,3)}{(3,3)},\frac{(4,4)}{(4,4)},\frac{(5,5)}{(2,2)}\right)}(x_1,\dots,x_8;q^{-1})$$

where $H=H(R_q,R_t,R_{qt},(2,2,1,1,0,0))=\sum_{\omega\in S_6}\omega\left(\frac{z_1^2z_2^2z_3z_4(1-qt\frac{z_1}{z_5})(1-qt\frac{z_1}{z_6})(1-qt\frac{z_2}{z_5})(1-qt\frac{z_2}{z_6})}{\prod_{1\le i\le 8}(1-\frac{z_i}{z_i})\prod_{(i,j)\in R}(1-q\frac{z_i}{z_j})(1-t\frac{z_i}{z_j})}\right)$ where $R=\{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,5),(3,6),(4,5),(4,6)\}$

e.g. $n=4$, $p=0.5+\varepsilon$, $\vec{d}=(3,2,2,1,-1)$, $\vec{e}=(1,2,2,1,1)$, $\vec{g}=(2,2,2,2)$ $\quad d=|\vec{g}|=8$. $(d_0-e_1, d_1-e_2, d_2-e_3, d_3-e_4)$





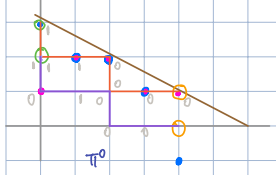
$R_q = R_t =$

$H = H(R_q, R_t, R_{qt}, (1,1,0,0,1,1,0,0))$

* In this case, H_{pol} is the same as $H(R_q', R_t', R_{qt}', (1,0,0,1,0))$ where R_q', R_t', R_{qt}' are obtained by removing the last two columns of R_q, R_t, R_{qt} respectively.

Note: $c_5<c_3<c_1<c_4<c_2$ $\therefore \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$

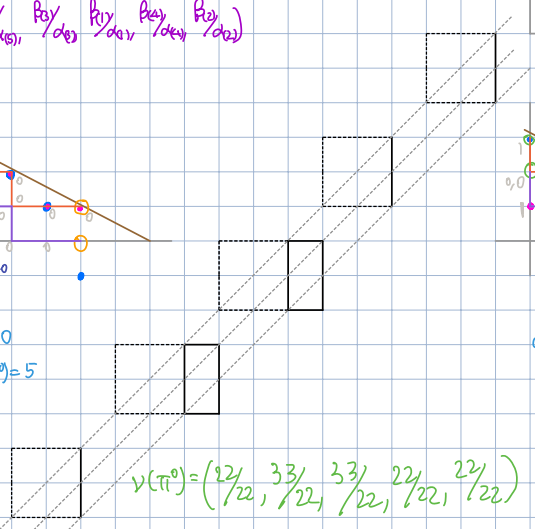
$\nu(\pi) = \left(R_{\sigma(5)}/d_{\sigma(5)}, R_{\sigma(3)}/d_{\sigma(3)}, R_{\sigma(1)}/d_{\sigma(1)}, R_{\sigma(4)}/d_{\sigma(4)}, R_{\sigma(2)}/d_{\sigma(2)} \right)$



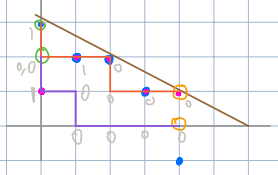
π^0

$a(\pi^0)=0$

$\mathrm{dinv}_p(\pi^0)=5$

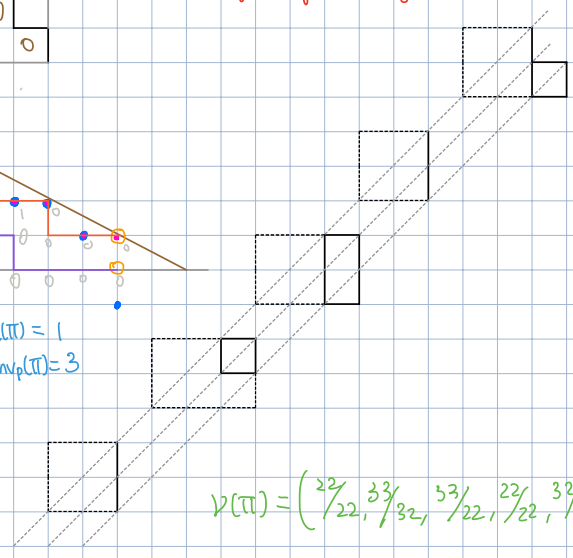


$\nu(\pi^0) = \left(22/22, 33/22, 33/22, 22/22, 22/22 \right)$



$a(\pi) = 1$

$\mathrm{dinv}_p(\pi) = 3$



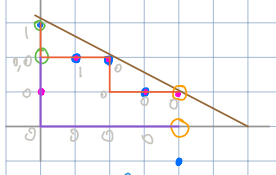
$\nu(\pi) = \left(22/22, 33/32, 33/22, 22/22, 32/22 \right)$

$t^0 q^5 \mathcal{G}(22/22, 33/22, 33/22, 22/22, 22/22)(z;q^{-1})$

$= t^0 q^5 \left(q^{-1} S_{22}(z) + q^{-2} S_{211}(z) + q^{-3} S_{1111}(z) \right)$

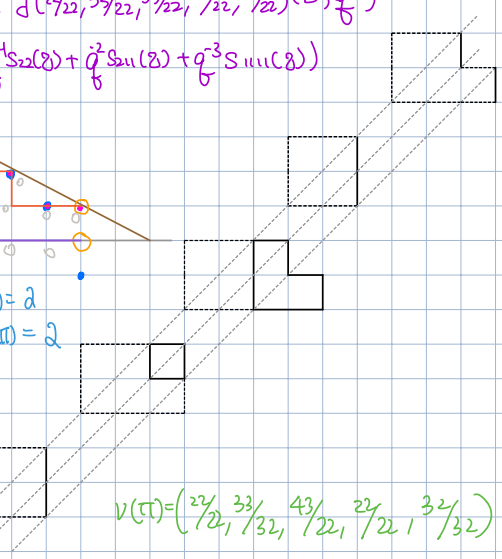
$t q^3 \mathcal{G}(22/22, 33/32, 33/22, 22/22, 32/22)(z;q^{-1})$

$= t q^3 \left(S_{31}(z) + S_{22}(z) + 2q^{-1} S_{211}(z) + q^{-2} S_{1111}(z) \right)$

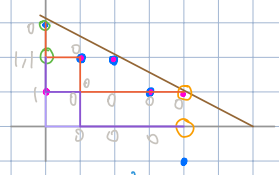


$a(\pi) = 2$

$\mathrm{dinv}_p(\pi) = 2$

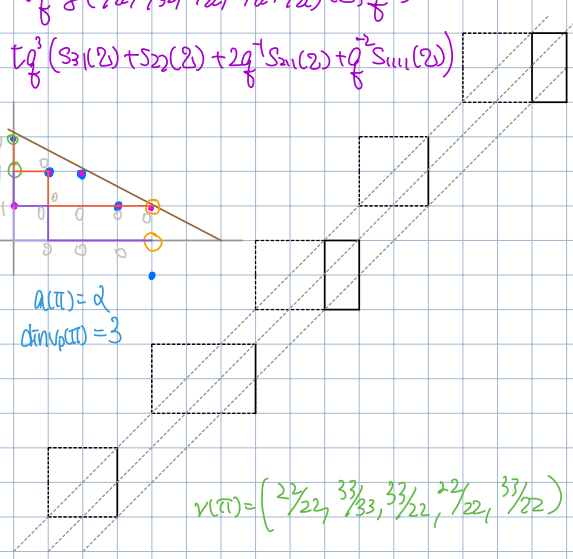


$\nu(\pi) = \left(22/22, 33/32, 43/22, 22/22, 32/32 \right)$



$a(\pi) = 2$

$\mathrm{dinv}_p(\pi) = 3$



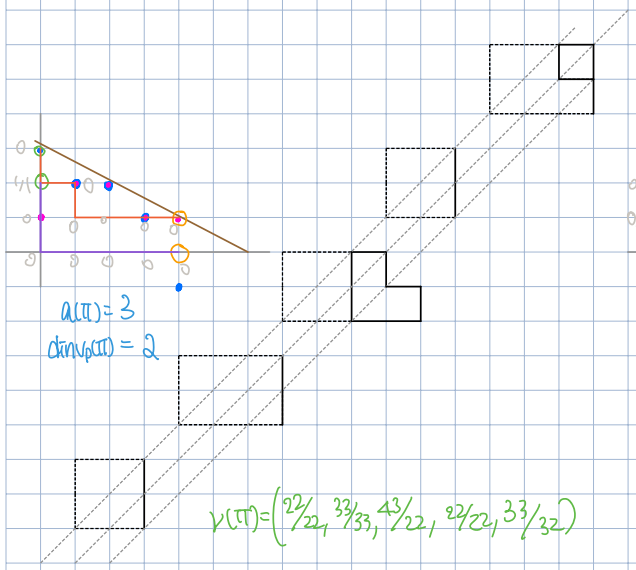
$\nu(\pi) = \left(22/22, 33/33, 33/22, 22/22, 33/22 \right)$

$t^2 q^2 \mathcal{G}(22/22, 33/32, 43/22, 22/22, 32/32)(z;q^{-1})$

$= t^2 q^2 \left(S_{31}(z) + S_{22}(z) + q^{-1} S_{211}(z) \right)$

$t^2 q^3 \mathcal{G}(22/22, 33/33, 33/22, 22/22, 33/22)(z;q^{-1})$

$= t^2 q^3 \left(q^{-1} S_{22}(z) + q^{-2} S_{211}(z) + q^{-3} S_{1111}(z) \right)$



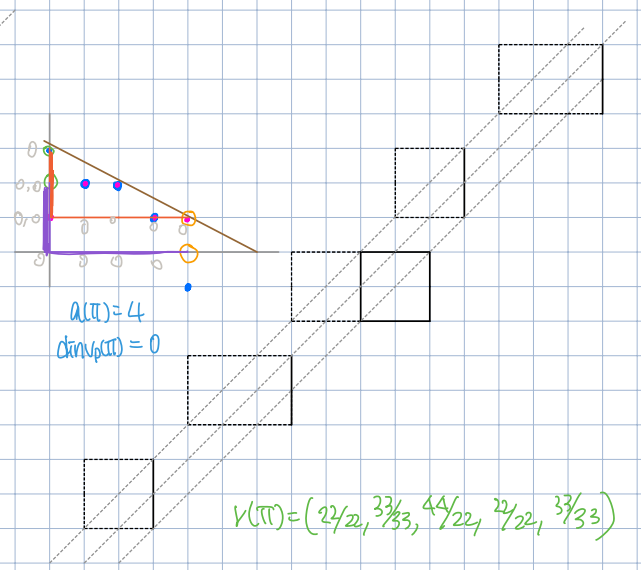
$a(\Pi) = 3$

$\mathrm{dinv}_p(\Pi) = 2$

$\nu(\Pi) = \left(22/22, 33/33, 43/22, 22/22, 33/32\right)$

$t^3 q^2\, \mathcal{G}(22/22, 33/32, 43/22, 22/22, 33/32)(z;q^{-1})$

$= t^3 q^2 \left(q^{-1} s_{31}(z) + q^{-1} s_{22}(z) + q^{-2} s_{211}(z)\right)$



$a(\Pi) = 4$

$\mathrm{dinv}_p(\Pi) = 0$

$\nu(\Pi) = \left(22/22, 33/33, 44/22, 22/22, 33/33\right)$

$t^4 q^0\, \mathcal{G}(22/22, 33/33, 44/22, 22/22, 33/33)(z;q^{-1})$

$= t^4 q^0 s_{22}(z)$

H_{pol} = sum of all these 6 LLT $\cdot\, t^{a(\Pi)} q^{\mathrm{dinv}_p(\Pi)}$

$= (q^3 t + q^2 t^2 + q t^3) s_{31}(z) + (q^4 + q^3 t + 2q^2 t^2 + q t^3 + t^4) s_{22}(z) + (q^3 + 2q^2 t + 2q t^2 + t^3) s_{211}(z) + (q^2 + q t + t^2) s_{1111}(z).$